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On Mathematical Modelling of Time-Related Musical Structures

Abstract. In this article, the first steps towards mathematical modelling of time-related musical structures are taken, and the algebraic structure of musical time relations is elaborated starting from a perceptive point of view. A basic characterization of fundamental properties of perceived time relations and their interpretations regarding musical context are given, and some mathematical properties of the proposed definitions are examined. Stemming from musical motivation a category is found whose objects are finite strict (partially) ordered sets and whose morphisms are weakly monotone and reflect the strict order of the codomain. The category is found to have initial and terminal objects, equalizers, and coequalizers but fails to have binary products or coproducts.

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On Mathematical Modelling of Time-Related Musical Structures

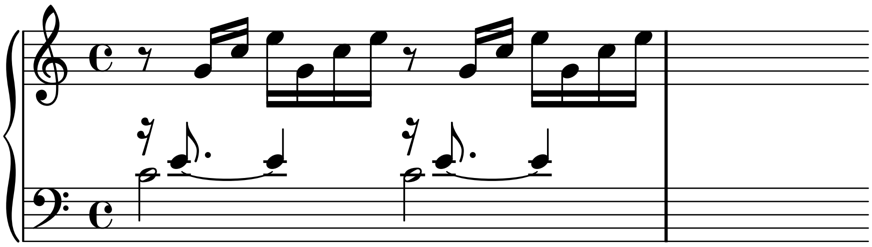
In order to model time-related musical structures it is essential to first model the perception of time to some useful extent. In [4] via [3] it is stated that time perception consists of *»the experience of (i) duration; (ii) non-simultaneity; (iii) order; (iv) past and present; (v) change, including the passage of time.«* From an algebraic point of view the most prominent among these appears to be order, and in musical context, perceived non-simulateneity implies perceived order in almost all cases.[1]

Definition 1. A pair $(T, <)$ is called *musical time relation*, if T is a finite set and $<\subseteq T \times T$ such that $<$ is a strict (partial) order, i.e. $<$ is an irreflexive and transitive relation, i.e. for all $x, y, z \in T$: $x \not< x$, and $x < y \wedge x < z \Rightarrow x < z$.

A musical time relation is then to be interpreted as a set of distinct musical events T, such that two elements $s, t \in T$ are ordered by $<$, i.e. $s < t$, if the end of the event corresponding to s is perceived before or at the same time the start of the event corresponding to t is perceived. In many cases, $s \not< t \not< s$ can be interpreted in such a way, that the events corresponding to s resp. t are perceived both for at least some small amount of time.

Example. Take a look at the first measure of J.S. Bach's Prelude No. 1 in C major (BWV 846):

[1] In fact it is possible to perceive distinct events as non-simultaneous without perception of a particular order, but the time scales where this occurs are very short and thus of lesser interest from the musical point of view.



One could identify all played notes as the core musical events that make up the piece. Then it is easy to obtain the underlying musical time relation of the notes of the first measure by first giving each note a unique name in order to make up the set T, e.g. by counting them in the order of appearance[2]

$$T = \{1, 2, 3, \dots, 16\}$$

and afterwards obtaining the relation $<$ by asking for $s, t \in T$, whether s ends before or at the start of t. Here $<$ is the following transitive closure:

$$\begin{aligned} < \quad = \quad & \text{trans}\{(1,9),(2,9),(3,4),(4,5),(5,6),(6,7),(7,8),(8,9) \\ & (11,12),(12,13),(13,14),(14,15),(15,16)\} \end{aligned}$$

Definition 2. Let $(T, <)$ be a musical time relation. $G \subseteq T$ is called *coarse grouping* wrt. $(T, <)$, if for all $x \in T \backslash G$ and all $g, h \in G$ the equivalences

$$x < g \;\Leftrightarrow\; x < h \,\text{and}\, g < x \;\Leftrightarrow\; h < x$$

hold.

A coarse grouping consists of distinct events, which relate in the same way towards the elements that are not part of the coarse grouping, and thus can be considered to form a single complex event.

Example. Consider again our motivational example, the first measure of the Prelude No. 1. Musical experience suggests, that when listening to the actual piece, the core musical events are perceived grouped into phrases and chords, and that there are different levels of grouping, where the finest level consists of the core events and the coarsest level consists of the piece as a whole.

Such grouping intuitively is done in a way, such that the events grouped form blocks of simultaneous and/or successional events, i.e. there are no gaps consisting of grouping-external events in between. Furthermore, the events grouped should possess the same time characteristics towards grouping-external events: In the example, it would be perfectly valid to have a grouping consisting of $\{1, 2\}$, since these notes form an arpeggiated chord and have the same time relation both towards the second half of the measure and the discant arpeggio notes in the first half. Further $\{1, 2, 3, 4, 5, 6, 7, 8\}$ would form a valid grouping since it makes up the whole first half of the first measure, $\{3, 4, 5\}$ would be a valid grouping since it consists of a melodic phrase, whereas $\{1, 2, 3, 4, 5\}$ would not make a valid grouping, since notes 1 and 2 are not perceived non-simultaneous with the notes 6, 7, and 8; but 3, 4, and 5 are. Therefore coarse groupings and the act of coarsening have some natural musical meaning, which motivates the further mathematical investigation of these structures.

Definition 3. Let $(T, <)$ and $(S, \sqsubset)$ be musical time relations. A map $\phi\colon T \to S$ is called *coarsening map* between $(T, <)$ and $(S, \sqsubset)$, if ϕ is surjective, weakly monotone, i.e. for $x, y \in T$:

$$x < y \;\Rightarrow\; \phi x \sqsubset \phi y \,\vee\, \phi x = \phi y$$

[2]Notice that the tied e′s are a single notes using two stems in notation.

and if ϕ reflects $\sqsubset$, i.e. for $x, y \in S$:

$$\phi^{-1}x \times \phi^{-1}y \subseteq < \Leftrightarrow x \sqsubset y$$

where $\phi^{-1}x = \{t \in T \mid \phi t = x\}$.

Lemma 4. *Let $(T, <)$ and $(S, \sqsubset)$ be musical time relations, and $\phi\colon T \to S$ a coarsening map. Then every equivalence class of the kernel of ϕ, i.e. $\phi^{-1}s$ for some $s \in S$, is a coarse grouping wrt. $(T, <)$.*

Proof. Let $g, h \in \phi^{-1}s$ and $x \in T \backslash \phi^{-1}s$. Then there is some $r \in S$ with $s \neq r$ such that $x \in \phi^{-1}r$.

$$x < g \Rightarrow r \sqsubset s \Rightarrow \phi^{-1}r \times \phi^{-1}s \subseteq < \Rightarrow x < h$$

Furthermore let $x \not< g$ and assume $x < h$, then $\phi x \sqsubset \phi h$, i.e.

$$r \sqsubset s \Rightarrow \phi^{-1}r \times \phi^{-1}s \subseteq < \Rightarrow x < g$$

in contradiction to $x \not< g$. Thus $x < g \Leftrightarrow x < h$ holds. Analogously, $g < x \Leftrightarrow h < x$ holds, and so $\phi^{-1}s$ is a coarse grouping. □

Definition 5. Let $(T, <)$ be a musical time relation and $\gamma \subseteq 2^T$ a partition of T, i.e. for all $G, H \in \gamma$ either $G = H$ or $G \cap H = \emptyset$ holds; $\bigcup \gamma = T$, and $\emptyset \notin \gamma$. γ is called *coarsening partition* wrt. $(T, <)$ if all $G \in \gamma$ are coarse groupings wrt. $(T, <)$.

Lemma 6. *Let $(T, <)$ be a musical time relation and $\gamma \subseteq 2^T$ a coarsening partition wrt. $(T, <)$. Then the following map is a coarsening map wrt. $(T, <)$ and $(S, \sqsubset)$ for $S = \gamma$ and $\sqsubset = \{(G, H) \in S \times S \mid G \times H \subseteq <\}$:*

$$\phi\colon T \to \gamma,\ x \mapsto G_x \text{ with } x \in G_x$$

Proof. Since γ is a partition of T, ϕ is surjective. Furthermore, let $x < y$ and $\phi x \neq \phi y$. Since ϕx is a coarse grouping we have for all $g \in \phi x$: $g < y$ and analogously we have for all $h \in \phi y$: $x < h$, and by iteration of this argument, $g < h$ for all $g \in \phi x$ and $h \in \phi y$, thus $\phi x \times \phi y \subseteq <$. This means $x < y \Rightarrow \phi x \sqsubset \phi y$, i.e. ϕ is weakly monotone. ϕ obviously reflects $\sqsubset$ by definition. □

For a given coarsening partition γ, the map ϕ from lemma 6 will be denoted by $[\bullet]_\gamma$ and ϕx will be denoted by $[x]_\gamma$.

Lemma 7. *Let $(T, <)$ be a musical time relation and $\alpha, \beta \subseteq 2^T$ both coarsening partitions wrt. $(T, <)$. Then*

$$\gamma = \{A \cap B \mid A \in \alpha,\ B \in \beta\} \setminus \{\emptyset\}$$

is a coarsening partition wrt. $(T, <)$.

Proof. Let $A \in \alpha$ and $B \in \beta$ such that $A \cap B \neq \emptyset$. Then for all $g, h \in A \cap B$ and $x \in T \backslash A$, the equalities

$$g < x \Leftrightarrow h < x$$

and

$$x < g \Leftrightarrow x < h$$

hold since A is a coarse grouping, analogously for $x \in T \backslash B$. Thus the equalities also hold for $x \in T \backslash (A \cap B)$, and the partition γ is indeed a coarsening partition wrt. $(T, <)$. □

Not only within musical contexts, time is often modelled in such a way that it resembles a time bar, for example by modelling time points as elements of the linear ordered set $(\mathbb{R}, \leq)$. Such modelling of start resp. end points of events in time has certain impact on the observed structure of the resultant musical time relations.

Definition 8. A musical time relation $(T,<)$ is called *special*, if for all $j,k,l,m \in T$ the implication

$$j < k \wedge l < m \Rightarrow j < m \vee l < k$$

holds.

Theorem 9. *Let $(T,<)$ be a musical time relation. $(T,<)$ is special, if and only if there are maps $\sigma, \tau \colon T \to \mathbb{R}$ such that*

$$x < y \Leftrightarrow \tau x \leq \sigma y$$

Remark. Since $<$ is irreflexive, it is clear that for all $x \in T$ the maps must satisfy the strict unequality $\sigma x < \tau x$.

Proof. Let σ, τ be such maps. Then for $j,k,l,m \in T$ with $j < k$ and $l < m$ we have $\tau j \leq \sigma k$ and $\tau l \leq \sigma m$. Since $\leq$ on $\mathbb{R}$ is linear, we have $\tau j \leq \tau l$ or $\tau l \leq \tau j$. We assume that $\tau j \leq \tau l$. Then $\tau j \leq \tau l \leq \sigma m$, i.e. $j < m$. Otherwise $\tau l \leq \tau j \leq \sigma k$, i.e. $l < k$. Conversely we will construct such a pair of maps as follows:[3]

$$\alpha \colon T \to \mathbb{R},\ x \mapsto \{w \in T \mid w < x\}$$

$$\beta \colon T \to \mathbb{R},\ x \mapsto \{w \in T \mid \forall z \in T \colon x < z \Rightarrow w < z\}$$

Assume that $x < y$, since $<$ is transitive, for all $t \in T\colon\ t < x \Rightarrow t < y$, i.e.

$$t \in \{w \in T \mid w < y\} \Rightarrow t \in \{w \in T \mid \forall z \in T \colon x < z \Rightarrow w < z\}$$

we have $\beta(x) \subseteq \alpha(y)$. Furthermore, let R be the set of the images of α and β, i.e.

$$R = \{\alpha x \mid x \in T\} \cup \{\beta x \mid x \in T\}$$

Then R is linearly ordered wrt. $\subseteq$: Assume there are $P, Q \in R$ such that $P \not\subseteq Q \not\subseteq P$, and let $p \in P$ with $p \notin Q$, and $q \in Q$ with $q \notin P$. Then there are $s,t \in T$ with $p < s$ and $q < t$, but still $p \not< t$ and $q \not< s$, which contradicts that $(T,<)$ is special: If P resp. R is in the image α, i.e. $P = \alpha x$ resp. $R = \alpha x$, choose x for s resp. t. Otherwise $P = \beta x$ resp. $R = \beta x$. Then choose s resp. t from the set $\{z \in T \mid x < z\}$ which is non-empty since $\beta x = \bigcap^T \{\alpha z \mid x < z\}$, and thus

$$\{z \in T \mid x < z\} = \emptyset \Rightarrow \beta x = T$$

If $\{z \in T \mid x < z\}$ is empty, you cannot have $P \not\subseteq Q \not\subseteq P$ since $P, Q \subseteq T$. Since R is linearly ordered, we can define the desired maps by counting the elements of α resp. β:

$$\sigma \colon T \to \mathbb{R},\ x \mapsto \#\alpha x$$

$$\tau \colon T \to \mathbb{R},\ x \mapsto \#\beta x$$

□

Whenever there is a binary relation, it can be interpreted in terms of formal concept analysis, a very brief introduction shall be given for the unfamiliar reader.

Definition 10. (See [2]). A tuple (G,M,I) is called a *formal context*, if G is a set whose elements are called *objects*, and M is a set whose elements are called *attributes*, and $I \subseteq G \times M$ is a relation, called the *incidence relation*.

Definition 11. (See [2]). Let (G,M,I) be a formal context. The formal concept lattice wrt. (G,M,I) is defined as the complete lattice

$$\mathcal{B}(G,M,I) = (P, \leq, \wedge, \vee, 0, 1)$$

where

$$P = \{(E'', E') \mid E \subseteq G\}$$

[3] This part of the proof follows an idea found in [1].

with

$$\bullet'': 2^G \to 2^G,\ E \mapsto \{g \in G \mid \forall m \in M\colon\ (\forall e \in E\colon e\, I\, m) \Rightarrow g\, I\, m\}$$

and

$$\bullet': 2^G \to 2^M,\ E \mapsto \{m \in M \mid \forall e \in E\colon e\, I\, m\}$$

Furthermore for $E, F \subseteq G$:

$$(E'', E') \leq (F'', F') \Leftrightarrow E'' \subseteq F'' \Leftrightarrow E' \supseteq F'$$

and

$$\bullet \wedge \bullet\colon P \times P \to P,\ ((E'', E'), (F'', F')) \mapsto \left(E'' \cap F'', (E'' \cap F'')'\right)$$

The elements of P are called *formal concepts* wrt. (G, M, I) and for $(E'', E') \in P$, E'' is called the *extent* of (E'', E') and E' is called the *intent* of (E'', E').

Definition 12. The *musical time context* of $(T, <)$ is defined as the formal context with objects and attributes both T and the incidence relation $<$, i.e. $(T, T, <)$.

Formal concepts of a musical time context can be interpreted to be transitional time spans or connecting time points between the end of the events of the extent and the start of the events of the intent. Therefore one can expect that a time bar modelling of event start and end points yields linear orders for resultant musical time context concept lattices:

Theorem 13. *Let $(T, <)$ be a special musical time relation, then the concept lattice of $(T, T, <)$ is linearly ordered.*

Proof. This follows easily from the proof of theorem 9: Let $P, Q \subseteq T$ be extents of formal concepts wrt. $(T, T, <)$, i.e. $P = P''$ and $Q = Q''$. If $P \not\subseteq Q \not\subseteq P$, then there are $p \in P \backslash Q$, $q \in Q \backslash P$, $s \in P' \backslash Q'$, and $t \in Q' \backslash P'$. But this yields $p < s$, $q < t$, $p \not< t$ and $q \not< s$ which contradicts that $(T, <)$ is special. □

Remark. In general, for some $(T, <)$ and $E, G \subseteq T$; G is a coarse grouping and $G = E'$ or $G = E''$ are independent properties.

The notion of coarsening maps give an idea of how surjective homomorphisms between musical time relations should look like. From a category theoretic point of view, surjective maps are very special and it is desirable to derive a notion of what general homomorphisms between musical time relations look like.

Definition 14. Let $(T, <)$ be a musical time relation and $S \subseteq T$. The *restriction of $(T, <)$ to S* is then defined as

$$(T, <)|S = (S, < \cap\, S \times S)$$

Definition 15. Let $(T, <)$ and $(S, \sqsubset)$ be musical time relations. A map $\phi\colon T \to S$ is called *weak coarsening map* between $(T, <)$ and $(S, \sqsubset)$, if the map

$$\psi\colon T \to \operatorname{im}\phi,\ x \mapsto \phi x$$

is a coarsening map between $(T, <)$ and $(S, \sqsubset)|\operatorname{im}\phi$.

In order to see that weak coarsening maps and the usual covariant function composition indeed form a category, we must show the following result:

Lemma 16. *Let $(T, <)$, $(S, \sqsubset)$, and $(R, \prec)$ be musical time relations, and let $\phi\colon T \to S$ and $\psi\colon S \to R$ be coarsening maps between $(T, <)$ and $(S, \sqsubset)$ resp. $(S, \sqsubset)$ and $(R, \prec)$. Then*

$$\phi * \psi\colon T \to R,\ x \mapsto \psi(\phi x)$$

is a coarsening map between $(T, <)$ and $(R, \prec)$.

Proof. Clearly, the composition of surjective maps is surjective. Further for all $x, y \in T$:

$$x < y \Rightarrow \phi x \sqsubset \phi y \vee \phi x = \phi y \Rightarrow \psi(\phi x) \prec \psi(\phi y) \vee \psi(\phi x) = \psi(\phi y)$$

thus $\phi * \psi$ is weakly monotone. For all $x, y \in R$ we have

$$x \prec y \Leftrightarrow \psi^{-1}x \times \psi^{-1}y \subseteq \sqsubset$$

and for all $s \in \psi^{-1}x$ and $t \in \psi^{-1}y$ we have

$$s \sqsubset t \Leftrightarrow \phi^{-1}s \times \phi^{-1}t \subseteq <$$

which yields

$$x \prec y \Leftrightarrow \psi^{-1}x \times \psi^{-1}y \subseteq \sqsubset \Leftrightarrow \phi * \psi^{-1}x \times \phi * \psi^{-1}y \subseteq <$$

thus $\phi * \psi$ is a coarsening map. □

Corollary 17. *Let $(T, <)$, $(S, \sqsubset)$, and $(R, \prec)$ be musical time relations, and let $\phi\colon T \to S$ and $\psi\colon S \to R$ be weak coarsening maps between $(T, <)$ and $(S, \sqsubset)$ resp. $(S, \sqsubset)$ and $(R, \prec)$. Then*

$$\phi * \psi\colon T \to R,\ x \mapsto \psi(\phi x)$$

is a weak coarsening map between $(T, <)$ and $(R, \prec)$.

Thus we can define the category of musical time relations as follows:

Definition 18. The *category of musical time relations* $\mathcal{M}$ has all musical time relations as objects, i.e.

$$\operatorname{Ob}\mathcal{M} = \{(T, <) \text{ musical time relation}\}$$

and weak coarsening maps as morphisms, i.e.

$$\operatorname{Mor}\mathcal{M} = \{((T, <), \phi, (S, \sqsubset)) \mid \phi \text{ weak coarsening map wrt. } (T, <) \text{ and } (S, \sqsubset)\}$$

The composition of compatible morphisms is given by

$$((T, <), \phi, (S, \sqsubset)) * ((S, \sqsubset), \psi, (R, \prec)) = ((T, <), \phi * \psi, (R, \prec))$$

and identity morphism for $(T, <) \in \operatorname{Ob}\mathcal{M}$ is

$$\mathrm{id}_{(\mathrm{T},<)} = ((T, <), \mathrm{id}_T, (T, <))$$

For $((T, <), \phi, (S, \sqsubset))$ we will write $\phi\colon (T, <) \to (S, \sqsubset)$ from now on.

Lemma 19. *$\mathcal{M}$ has terminal objects.*

Proof. Let $T \in \operatorname{Ob}\mathbf{Sets}$ such that $T = \{t\}$. Then $(T, \emptyset)$ is a terminal object in $\mathcal{M}$, since such T is a terminal object wrt. **Sets** and for $(S, \sqsubset) \in \operatorname{Ob}\mathcal{M}$ the unique morphism wrt. **Sets** is indeed $!_T^S\colon (S, \sqsubset) \to (T, \emptyset) \in \operatorname{Mor}\mathcal{M}$, since $(!_T^S)^{-1}t = S$ and thus $!_T^S$ is even a coarsening map. □

The existence of a terminal object in $\mathcal{M}$ can be interpreted in such a way, that every musical piece can be considered to form a single complex event.

Lemma 20. *$\mathcal{M}$ has an initial object.*

Proof. $(\emptyset, \emptyset)$ is the initial object of $\mathcal{M}$: Let $(T, <) \in \operatorname{Ob}\mathcal{M}$. Then $\mathrm{i}_T\colon (\emptyset, \emptyset) \to (T, <)$ is one and the only morphism between $(\emptyset, \emptyset)$ and $(T, <)$ since $\mathrm{i}_T = \phi$ is the only morphism in **Sets** with $\operatorname{dom}\phi = \emptyset$ and $\operatorname{cod}\phi = T$, and since $\operatorname{im}\mathrm{i}_T = \emptyset$, i_T clearly is a weak coarsening map. □

Lemma 21. *$\mathcal{M}$ has equalizers.*

Proof. Let $\phi, \psi \colon (T, <) \to (S, \sqsubset)$ be morphisms of $\mathcal{M}$. Then the set equalizer

$$\mathrm{eq}(\phi, \psi) \colon \mathrm{Eq}(\phi, \psi) \to T,\ x \mapsto x$$

with $\mathrm{Eq}(\phi, \psi) = \{\mathrm{t} \in \mathrm{T} \mid \phi\mathrm{t} = \psi\mathrm{t}\}$ is a weakly coarsening map between $(R, \prec)$ and $(T, <)$ with $R = \mathrm{Eq}(\phi, \psi)$ and $\prec = < \cap R \times R$, because id_R is a coarsening map between $(R, \prec)$ and $(R, \prec)$. Now let $\alpha \colon (S, \sqsubset) \to (T, <)$ be a morphism in $\mathcal{M}$ s.t. $\alpha * \phi = \alpha * \psi$. Then there is a unique map in **Sets**, such that $\alpha = \eta * \mathrm{eq}(\phi, \psi)$, with

$$\eta \colon S \to \mathrm{Eq}(\phi, \psi),\ x \mapsto \alpha x$$

The map η is a weakly coarsening map, since $(T, <)|\mathrm{im}\,\alpha = (R, \prec)|\mathrm{im}\,\eta$ and α is a weakly coarsening map. Thus

$$\mathrm{eq}(\phi, \psi) \colon (R, \prec) \to (T, <)$$

is the equalizer for ϕ and ψ. □

Theorem 22. *$\mathcal{M}$ has coequalizers.*

Proof. Let $\phi, \psi \colon (T, <) \to (S, \sqsubset)$ be morphisms of $\mathcal{M}$. There is a smallest coarsening partition $\gamma \subseteq 2^S$ with the property that for all $x \in T$ and $G \in \gamma$,

$$\phi x \in G \;\Leftrightarrow\; \psi x \in G$$

since there are only finitely many partitons of S, the following formula is sound

$$\gamma \quad = \quad \bigcap_{\beta \in C} \alpha$$

where

$$C = \left\{\beta \subseteq 2^S \mid \beta \text{ coars. part.}, \forall x \in T, G \in \beta \colon\ \phi x \in G \;\Leftrightarrow\; \psi x \in G\right\}$$

Then the projection map $[\bullet]_\gamma \colon (S, \sqsubset) \to (R, \prec)$ of the coarsening partition γ is the coequalizer of ϕ and ψ: Let $\alpha \colon (S, \sqsubset) \to (A, \lhd)$ s.t. $\phi * \alpha = \psi * \alpha$. The unique map for α is

$$\eta \colon (R, \prec) \to (A, \lhd),\ G \mapsto \alpha x \text{ with } G = [x]_\gamma$$

Let $x, y \in S$ with $[x]_\gamma = [y]_\gamma$. Then there is no coarse grouping $\beta \in C$ s.t. there is $G \in \beta$ with $x \in G \not\ni y$, therefore $\alpha x = \alpha y$, since

$$\text{eq ker } \alpha = \left\{\alpha^{-1} a \mid a \in A\right\}$$

is a coarsening partition and since for $x \in T$ we have $\alpha(\phi x) = \alpha(\psi x)$, it follows that eq ker $\alpha \in C$. Thus the definition of η is independent of the choice of the preimage of $[\bullet]_\gamma$. For $x, y \in S$,

$$[x]_\gamma \prec [y]_\gamma \;\Rightarrow\; x \sqsubset y \;\Rightarrow\; \alpha x \lhd \alpha y \,\vee\, \alpha x = \alpha y \;\Rightarrow\; \eta[x]_\gamma \lhd \eta[y]_\gamma \,\vee\, \eta[x]_\gamma = \eta[y]_\gamma$$

hence η is weakly monotone. Now let $a, b \in \mathrm{im}\,\alpha$.

$$\eta^{-1} a \times \eta^{-1} b \subseteq \prec \Leftrightarrow \left(\bigcup \eta^{-1} a\right) \times \left(\bigcup \eta^{-1} b\right) \subseteq \sqsubset$$

$$\Leftrightarrow\; \alpha^{-1} a \times \alpha^{-1} b \subseteq \sqsubset \Leftrightarrow\; a \lhd b$$

since $\left(\bigcup \eta^{-1} a\right) = \alpha^{-1} a$. Thus η is a weakly coarsening map. □

Proposition 23. *$\mathcal{M}$ fails to have binary coproducts.*

Proof. Let $T = \{1, 2\}$ and $< = \{(1, 2)\}$. There is no coproduct of $(T, <)$ and $(T, <)$ in $\mathcal{M}$. Assume $(R, \prec)$ is the coproduct of $(T, <)$ with $(T, <)$, and $\iota_1 \colon (T, <) \to (R, \prec)$ and $\iota_2 \colon (T, <) \to (R, \prec)$ are the two injection maps in $\mathcal{M}$. For $\alpha \colon (T, <) \to (A, \lhd)$ and $\beta \colon (T, <) \to (A, \lhd)$ the unique weakly coarsening map is denoted by $[\alpha, \beta]$, i.e. $\alpha = \iota_1 * [\alpha, \beta]$ and $\beta = \iota_2 * [\alpha, \beta]$. Consider the following weakly coarsening maps

$$\alpha_j \colon (T, <) \to (T, <),\ x \mapsto j \text{ for } j \in \{1, 2\}$$

Since $\alpha_1 x < \alpha_2 x$ for $x \in T$ and $[\alpha_1, \alpha_2]$ is a weakly coarsening map, we get from $1 < 2$

$$\{\iota_1 1, \iota_1 2\} \times \{\iota_2 1, \iota_2 2\} \subseteq [\alpha_1, \alpha_2]^{-1} 1 \times [\alpha_1, \alpha_2]^{-1} 2 \subseteq \prec$$

But since $[\alpha_2, \alpha_1]$ is also a weakly coarsening map, we also get

$$\{\iota_2 1, \iota_2 2\} \times \{\iota_1 1, \iota_1 2\} \subseteq [\alpha_2, \alpha_1]^{-1} 1 \times [\alpha_2, \alpha_1]^{-1} 2 \subseteq \prec$$

Then since $\prec$ is transitive

$$\iota_1 1 \prec \iota_2 2 \prec \iota_1 1 \;\Rightarrow\; \iota_1 1 \prec \iota_1 1$$

which contradicts that $\prec$ is irreflexive. □

Proposition 24. *$\mathcal{M}$ fails to have binary products.*

Proof. Consider $(T, <)$ from the proof of proposition 23. There is no product of $(T, <)$ with $(T, <)$ in $\mathcal{M}$. Assume that $(R, \prec)$ together with the projection maps $\pi_1 \colon (R, \prec) \to (T, <)$ and $\pi_2 \colon (R, \prec) \to (T, <)$ form the desired product in $\mathcal{M}$. The unique map for $\alpha \colon (A, \lhd) \to (T, <)$ and $\beta \colon (A, \lhd) \to (T, <)$ is denoted by $\langle \alpha, \beta \rangle$, i.e. $\alpha = \langle \alpha, \beta \rangle * \pi_1$ and $\beta = \langle \alpha, \beta \rangle * \pi_2$. Consider again $\alpha_1 \colon (T, <) \to (T, <)$ and $\alpha_2 \colon (T, <) \to (T, <)$ from the proof of proposition 23, and the identity map $\mathrm{id}_{(T,<)}$. Since for $x \in T$,

$$\left\langle \mathrm{id}_{(T,<)}, \alpha_1 \right\rangle * \pi_1 x = x$$

and

$$\left\langle \mathrm{id}_{(T,<)}, \alpha_1 \right\rangle * \pi_2 x = 1$$

we have

$$\pi_1^{-1} x \,\cap\, \pi_2^{-1} 1 \;\neq\; \emptyset$$

and analogously we get for $x, y \in T$ that

$$\pi_1^{-1} x \,\cap\, \pi_2^{-1} y \;\neq\; \emptyset$$

Further since $1 < 2$, we have

$$\pi_j^{-1} 1 \times \pi_j^{-1} 2 \subseteq \prec \text{ for } j = 1, 2$$

Let $x \in \pi_1^{-1} 1 \cap \pi_2^{-1} 2$ and $y \in \pi_1^{-1} 2 \cap \pi_2^{-1} 1$. Since π_1 is a weakly coarsening map, we have $x \prec y$, and since π_2 is a weakly coarsening map, we have $y \prec x$. Since $\prec$ is transitive, this yields $x \prec x$ and thus $\prec$ is not irreflexive. Therefore this particular binary product does not exists in $\mathcal{M}$. □

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